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Exact results in planar $\mathcal{N}=1$ superconformal Yang-Mills theory

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Abstract: In the β-deformed $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory we study the class of operators $\mathcal{O}_J = \mathrm{Tr}(\Phi_i^J \Phi_k)$, $i \neq k$ and compute their exact anomalous dimensions for $N, J \to \infty$. This leads to a prediction for the masses of the corresponding states in the dual string theory sector. We test the exact formula perturbatively up to two loops. The consistency of the perturbative calculation with the exact result indicates that in the planar limit the one-loop condition $g^2 = h\bar{h}$ for superconformal invariance is indeed sufficient to insure the *exact* superconformal invariance of the theory. We present a direct proof of this point in perturbation theory. The $\mathcal{O}_J$ sector of this theory shares many similarities with the BMN sector of the $\mathcal{N}=4$ theory in the large R-charge limit.

Keywords: Superspaces, AdS-CFT and dS-CFT Correspondence, Conformal and W Symmetry.

JHEP11(2005)024

Contents

1. Introduction

The recent past has seen a new interest on exactly marginal deformations of $\mathcal{N}=4$ SYM theory preserving $\mathcal{N}=1$ supersymmetry [1], in particular after the supergravity duals of the so-called β-deformations of $\mathcal{N}=4$ SU(N) SYM theory have been found by Lunin and Maldacena in [2].

Now new results start to emerge also from the field theory side: in [3 − 5] various properties of composite operators of the deformed theory have been investigated at the perturbative level (see also [6 − 9]). The outcome is that the deformed theory shares some properties with the undeformed $\mathcal{N}=4$ theory, but new features emerge, as for example the finite corrections of the two- and three-point functions of protected operators [4, 5].

The chiral ring of the theory was identified in [10, 11] for generic values of the deformation parameter. It is given by the operators $\mathrm{Tr}(\Phi_i^J)$, $i=1,2,3$, and $\mathrm{Tr}(\Phi_1^J\Phi_2^J\Phi_3^J)$. In [3, 4] it was shown that also the operator $\mathrm{Tr}(\Phi_1\Phi_2)$ does not acquire anomalous dimension.

In this paper we will focus on the operators $\mathrm{Tr}(\Phi_i^J\Phi_j)$, $i\neq j$. As opposed to what happens in the undeformed $\mathcal{N}=4$ case, these operators are not protected and their anomalous dimension was computed at one-loop in [3]. Our interest in these operators is motivated by the fact that in the large J limit they resemble the BMN operators of the $\mathcal{N}=4$ theory [12]. Indeed a perturbative superfield analysis performed at low orders shows that the supergraphs contributing to their anomalous dimension are exactly the same as the ones of the BMN case. Then we apply the derivation of [13] to this class of operators and compute their exact anomalous dimension in the planar limit.

The consistency between the perturbative supergraph approach and the result obtained by using the method of [13] suggests that the one-loop superconformal invariance condition remains valid in the planar limit to all orders of perturbation theory (at least for real values

JHEP11(2005)024

of the β parameter of the β-deformation which is the case we consider here). We confirm this result by exploiting the formal analogy between β-deformed and non-commutative field theories.

The paper is organized as follows. In section 2 we present the setup for the perturbative calculation that we perform in section 3, where we compute the anomalous dimensions of the operators $\mathrm{Tr}(\Phi_i^J \Phi_j)$, $i \neq j$ in the large-N limit up to order g^4. The contributing graphs are the same that one would encounter in the calculation of the anomalous dimension of BMN operators. In section 4 we apply to our operators the procedure introduced in [13] and give an all-order result for their anomalous dimensions in the large J and large-N limit. Then in section 5 we prove that the one-loop condition of superconformal invariance remains valid to all orders in the planar limit.

2. Generalities

We consider the following deformation of the $\mathcal{N} = 4$ SYM theory

$$\begin{aligned} S[j,\bar{j}] = &\int d^8 z \, \mathrm{Tr}\left(e^{-gV}\bar{\Phi}_i e^{gV}\Phi^i\right) + \frac{1}{2g^2}\int d^6 z \, \mathrm{Tr}\, W^\alpha W_\alpha + \\ &+ih\int d^6 z \, \mathrm{Tr}(q\Phi_1\Phi_2\Phi_3 - \frac{1}{q}\Phi_1\Phi_3\Phi_2) - \\ &-i\bar{h}\int d^6 \bar{z} \, \mathrm{Tr}(\bar{q}\bar{\Phi}_1\bar{\Phi}_3\bar{\Phi}_2 - \frac{1}{\bar{q}}\bar{\Phi}_1\bar{\Phi}_2\bar{\Phi}_3) + \int d^6 z \, j\mathcal{O} + \int d^6 \bar{z} \bar{j}\bar{\mathcal{O}} \end{aligned} \tag{2.1}$$

where we have set $q \equiv e^{i\pi\beta}$ and in the following we choose β real so that $q\bar{q} = 1$. We have added to the classical action source terms for composite chiral operators generically denoted by $\mathcal{O}$ with j $(\bar{j})$ (anti)chiral sources. The superfield strength $W_\alpha = i\bar{D}^2(e^{-gV}D_\alpha e^{gV})$ is given in terms of a real prepotential V and $\Phi_{1,2,3}$ contain the six scalars of the original $\mathcal{N} = 4$ SYM theory organized into the $\mathbf{3} \times \bar{\mathbf{3}}$ of SU(3) $\subset$ SU(4). We write $V = V^a T_a$, $\Phi_i = \Phi_i^a T_a$ where T_a are SU(N) matrices in the fundamental representation. In the following we use the notation $\mathrm{Tr}(T^a T^b T^c \ldots) \equiv (abc\ldots)$.

The β-deformation breaks the original SU(4) R-symmetry to U(1)$_R$. However, besides the Z_3 symmetry associated to cyclic permutations of (Φ_1, Φ_2, Φ_3), two extra non-R–symmetry U(1)'s survive. Applying the a-maximization procedure [14] and the conditions of vanishing ABJ anomalies it turns out that U(1)$_R$ is the one which assigns the same R-charge ω to the three elementary superfields, whereas the charges respect to the two non-R–symmetries U(1)$_1$ $\times$ U(1)$_2$ can be chosen to be $(\Phi_1, \Phi_2, \Phi_3) \to (0, 1, -1)$ and $(-1, 1, 0)$, respectively.

As discussed in [3, 4], at the quantum level the theory is superconformal invariant (and then finite) up to two loops if the coupling constants satisfy the following condition (vanishing of the beta functions)

$$h\bar{h}\left[1 - \frac{1}{N^2}\left|q - \frac{1}{q}\right|^2\right] - g^2 = 0\,. \tag{2.2}$$

In the large-N limit this condition reduces simply to $g^2 = h\bar{h}$, independently of the value of q. In [5] the three loop correction to (2.2) has been also evaluated. Since it turns out to be suppressed for $N \to \infty$, the condition $g^2 = h\bar{h}$ is the correct condition for superconformal invariance in the planar limit up to three loops.

At the superconformal fixed point of the theory we compute the anomalous dimensions for the class of non-protected operators[1]

$$\mathcal{O}_J = \mathrm{Tr}(\Phi_1^J \Phi_2)\,. \tag{2.3}$$

They are charged under $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ with charges $(1, 1-J)$.

Using the equations of motion from the action (2.1) with $j = \bar{j} = 0$ (from now on we neglect factors of $e^{\pm gV}$ since they are not relevant to our purposes)

$$\bar{D}^2 \bar{\Phi}_3^a = -ih\Phi_1^b \Phi_2^c [q(abc) - \frac{1}{q}(acb)] \tag{2.4}$$

it is easy to see that

$$\mathcal{O}_J = \frac{i}{h[q-\frac{1}{q}]} \bar{D}^2\, \mathrm{Tr}(\Phi_1^{J-1}\bar{\Phi}_3) + \frac{1}{N}\,\mathrm{Tr}(\Phi_1^{J-1})\,\mathrm{Tr}(\Phi_1\Phi_2)\,. \tag{2.5}$$

As long as $J > 1$, in the large-N limit the operator $\mathcal{O}_J$ becomes descendent of the primary $\mathrm{Tr}(\Phi_1^{J-1}\bar{\Phi}_3)$, whereas for finite N the combination $\mathcal{O}_J - \frac{1}{N}\mathrm{Tr}(\Phi_1^{J-1})\,\mathrm{Tr}(\Phi_1\Phi_2)$ is descendent. The exceptional case $J = 1$ corresponds to the chiral primary operator whose protection has been proven perturbatively in [3, 4].

In the next sections we will concentrate on the evaluation of the anomalous dimensions for the $\mathcal{O}_J$ operators.

3. The perburbative calculation

In this section we compute the anomalous dimension of $\mathcal{O}_J$ in (2.3) perturbatively, up to two loops. For generic values of J we perform the calculation in the large-N limit in order to avoid dealing with mixing with multitrace operators.

In order to compute anomalous dimensions we evaluate one-point correlators $\langle \mathcal{O}_J e^{S_{\mathrm{int}}}\rangle$ where S_{int} is the sum of the interaction terms in (2.1). Divergent contributions proportional to the operator itself are removed by a multiplicative renormalization which in dimensional regularization reads

$$\mathcal{O}_J^{(bare)} \equiv \mathcal{O}_J \left(1 + \sum_{k=0}^{\infty} \frac{a_k(\lambda, q, N)}{\epsilon^k} \right) \equiv Z\mathcal{O}_J \tag{3.1}$$

where we have introduced the 't Hooft coupling $\lambda = \frac{g^2 N}{4\pi^2}$. There is no dependence on the h coupling since we are at the superconformal point where h can be expressed in terms of the other couplings through the condition of vanishing beta functions.

The anomalous dimension is then given by

$$\gamma \equiv 2\lambda \frac{da_1(\lambda, q, N)}{d\lambda}\,. \tag{3.2}$$

[1]The choice of Φ_1 and Φ_2 superfields is totally arbitrary and we expect the operators $\mathrm{Tr}(\Phi_i^J\Phi_k)$, for any i, k with $i \neq k$ to have similar quantum properties. We will comment on this point later on.

JHEP11(2005)024

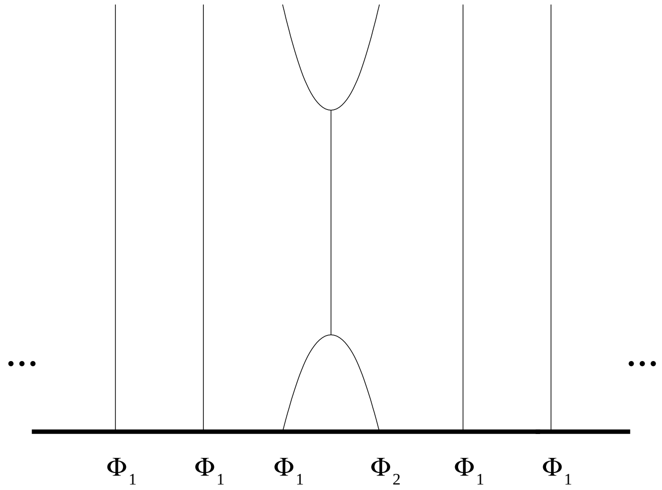

Figure 1: One-loop contribution to the $\mathcal{O}_J$ anomalous dimension.

Therefore, at any order it is easily read from the simple pole divergence.

We perform perturbative calculations in a superspace setup by following closely the procedure used in [15 − 18, 4] (we refer to those papers for conventions and technical details). After D-algebra supergraphs are reduced to ordinary Feynman diagrams which we evaluate in momentum space. We work in dimensional regularization, $d = 4 - 2\epsilon$, and minimal subtraction scheme.

From the action (2.1) quantized in the Feynman gauge we read the superfield propagators ($z \equiv (x, \theta, \bar{\theta})$)

$$\begin{aligned} \langle V^a(z_1) V^b(z_2) \rangle &= \delta^{ab} \frac{1}{(x_1 - x_2)^2} \delta^{(4)}(\theta_1 - \theta_2) \\ \langle \Phi_i^a(z_1) \bar{\Phi}_j^b(z_2) \rangle &= -\delta_{ij} \delta^{ab} \frac{1}{(x_1 - x_2)^2} \delta^{(4)}(\theta_1 - \theta_2) \end{aligned} \tag{3.3}$$

and the three-point vertices

$$\begin{aligned} (\Phi\Phi\Phi)_{\text{vertex}} &\to ih \Phi_1^a \Phi_2^b \Phi_3^c \left[q(abc) - \frac{1}{q}(acb) \right] \\ (\bar{\Phi}\bar{\Phi}\bar{\Phi})_{\text{vertex}} &\to -i\bar{h} \bar{\Phi}_1^a \bar{\Phi}_2^b \bar{\Phi}_3^c \left[\bar{q}(acb) - \frac{1}{\bar{q}}(abc) \right] \\ (\bar{\Phi} V \Phi)_{\text{vertex}} &\to g \bar{\Phi}_i^a V^b \Phi_i^c \left[(abc) - (acb) \right]. \end{aligned} \tag{3.4}$$

At the lowest order the only contribution to the one-point function for the operator $\mathcal{O}_J$ is the one given in figure 1 where, using the notation introduced in [19], the horizontal bold line indicates the spacetime point where the operator is inserted.

The corresponding contribution is proportional to the self-energy integral

$$I_1 \equiv \int d^n k \frac{1}{k^2 (p-k)^2} \sim \frac{1}{(4\pi)^2} \frac{1}{\epsilon}. \tag{3.5}$$

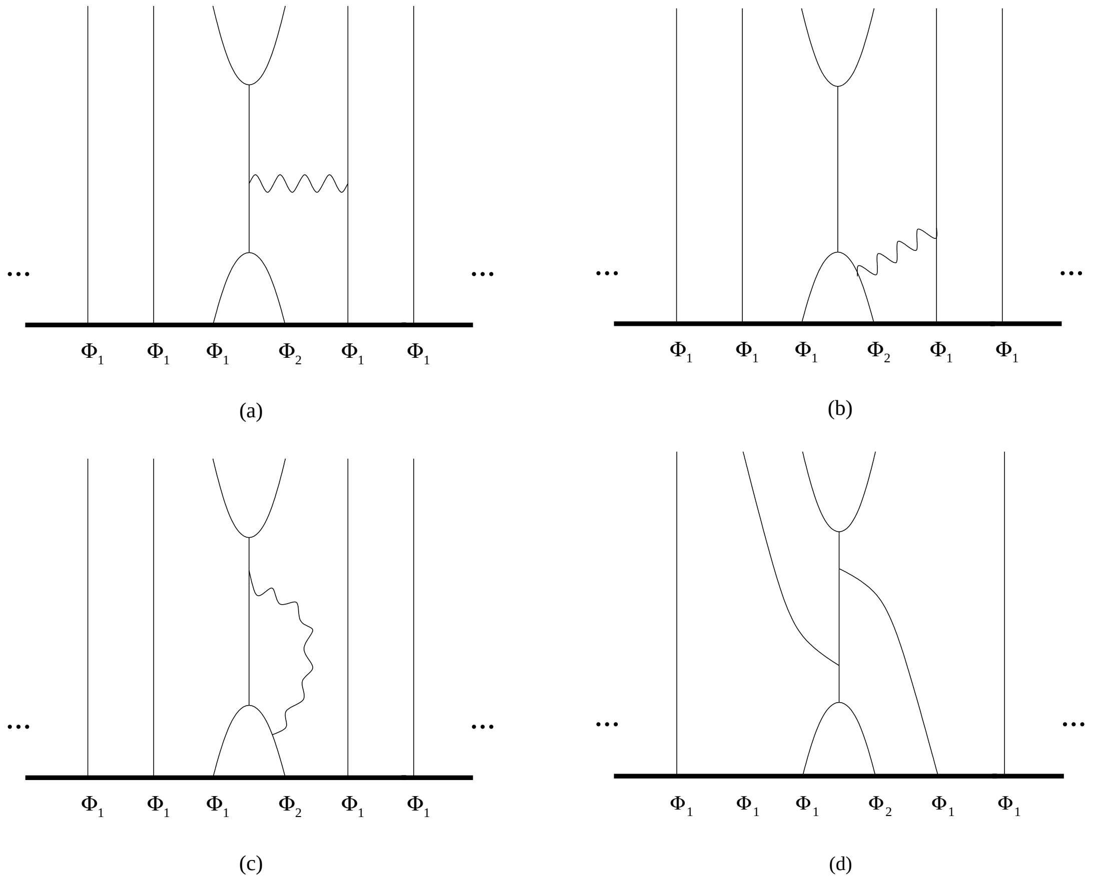

Figure 2: Two-loop contributions to the $\mathcal{O}_J$ anomalous dimension.

Evaluating the color factor, the combinatorics and taking into account a minus sign from D-algebra we obtain

$$\text{Diagram } 1 \to -\frac{1}{\epsilon}\left|q-\frac{1}{q}\right|^2 \frac{|h|^2 N}{(4\pi)^2}\,. \tag{3.6}$$

Using the one-loop superconformal condition in the planar limit ($g^2 = |h|^2$) and the definition (3.2) we immediately find the one-loop anomalous dimension

$$\gamma^{(1)} = \frac{1}{2}\left|q-\frac{1}{q}\right|^2 \lambda\,. \tag{3.7}$$

At two loops (order λ^2) the diagrammatic contributions are drawn in figure 2.

Performing the D-algebra we reduce all the diagrams to ordinary Feynman diagrams containing the loop structure as in figure 3.

The associated momentum integral is

$$I_2 \equiv \int d^n k_1 d^n k_2 \frac{1}{k_1^2(p_1-k_1-k_2)^2 k_2^2(p_1+p_2-k_2)^2}\,. \tag{3.8}$$

As long as we are only concerned with UV divergences we can safely set one of the external momenta to zero. Thus the graph is easily evaluated being proportional to two nested

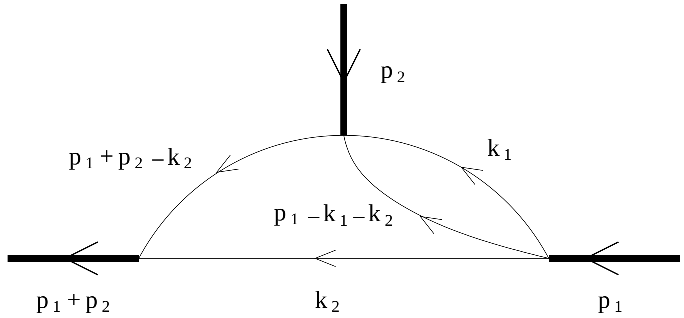

Figure 3: The two-loop Feynman diagram.

self-energies. We obtain (in the G-scheme)

$$I_2 \sim \frac{1}{(4\pi)^4}\frac{1}{2\epsilon^2}(1+5\epsilon)\frac{1}{(p^2)^{2\epsilon}} \tag{3.9}$$

where we have kept only divergent terms. Performing the subtraction of the subdivergence we finally have

$$[I_2]_{sub} \sim \frac{1}{(4\pi)^4}\left[-\frac{1}{2\epsilon^2}+\frac{1}{2\epsilon}\right]. \tag{3.10}$$

Computing the combinatorics, the color factors and taking into account minus signs from the vector propagator we find that the factors in front of (3.10) for the various diagrams are

$$\begin{aligned}
(2a) &\rightarrow -2\left(q-\frac{1}{q}\right)\left(\bar{q}-\frac{1}{\bar{q}}\right)g^2|h|^2N^2\\
(2b) &\rightarrow 2\left(q-\frac{1}{q}\right)\left(\bar{q}-\frac{1}{\bar{q}}\right)g^2|h|^2N^2\\
(2c) &\rightarrow 2\left(q-\frac{1}{q}\right)\left(\bar{q}-\frac{1}{\bar{q}}\right)g^2|h|^2N^2\\
(2d) &\rightarrow -\left(q-\frac{1}{q}\right)\left(\bar{q}-\frac{1}{\bar{q}}\right)\left(\frac{q}{\bar{q}}+\frac{\bar{q}}{q}\right)|h|^4N^2\,.
\end{aligned} \tag{3.11}$$

Summing all the contributions, using the planar superconformal condition $|h|^2 = g^2$ and the definition (3.2), we find

$$\gamma^{(2)} = -\frac{1}{8}\left|q-\frac{1}{q}\right|^4\lambda^2\,. \tag{3.12}$$

We observe that the diagrams contributing to the anomalous dimensions for our operators are exactly the same as the ones for BMN operators in $\mathcal{N}=4$ SYM in the planar limit [12, 19]. In fact, up to this order the calculation is exactly the same under the formal identification $|q-\frac{1}{q}|^2 \leftrightarrow -(e^{i\phi}+e^{-i\phi}-2)$, where ϕ is the phase of BMN operators [12, 19, 13]. We expect that the same pattern will persist at any order in perturbation theory. In particular, as in the BMN case, the graphs relevant for the calculation are only the ones where the interactions are close to the "impurity" Φ_2: at L-loop order the interactions may involve at most the Φ_1 lines which are L-steps far away from the impurity. As an important consequence, in the large J limit the anomalous dimensions do not grow with J.

JHEP11(2005)024

To close this section we note that the result we have found for the anomalous dimensions of the operators $\mathrm{Tr}(\Phi_1^J\Phi_2)$ at large-N are actually valid for any operator of the form $\mathrm{Tr}(\Phi_i^J\Phi_k)$ with $i \neq k$. In fact the superpotential is invariant under cyclic permutation of (Φ_1, Φ_2, Φ_3), and in addition it becomes invariant if non-cyclic exchanges of fields are accompanied by

$$q \to -\frac{1}{q}\,. \tag{3.13}$$

Since the anomalous dimensions are proportional to powers of the effective coupling $\alpha \equiv \lambda\left|q - \frac{1}{q}\right|^2$ which is invariant under (3.13) we conclude that the result is valid for any operator of the form $\mathrm{Tr}(\Phi_i^J\Phi_k)$, $i \neq k$.

4. The exact anomalous dimensions

Motivated by the formal correspondence of the previous calculation with the BMN case, in this section we are going to compute the *exact* anomalous dimensions in the large-N, large J limit by using the procedure introduced in [13] for BMN operators. In the context of β-deformed theories the same procedure has been applied to the class of BMN operators [6].

We concentrate on the operator $\mathcal{O}_{J+1}$ which, as follows from eq. (2.5), in the planar limit satisfies

$$\bar{D}^2\mathcal{U}_J = -ih[q - \frac{1}{q}]\mathcal{O}_{J+1} \tag{4.1}$$

where we have defined

$$\mathcal{U}_J \equiv \mathrm{Tr}(\Phi_1^J\bar{\Phi}_3)\,. \tag{4.2}$$

As already noticed, this shows that the $\mathcal{O}_{J+1}$ operators are descendants of the $\mathcal{U}_J$ ones. Being part of the same superconformal multiplet they share the renormalization properties, i.e. they will have the same scaling dimension and the same perturbative corrections to their overall normalization. Moreover since $\mathcal{U}_J$ is not a Konishi-like operator it is not affected by the Konishi anomaly.

As discussed in details in [13], in any $N = 1$ superconformal field theory the two-point function for a *primary* operator $\mathcal{A}_{s,\bar{s}}$ is fixed [20] and given by $(z \equiv (x, \theta, \bar{\theta}))$

$$\begin{aligned}\left\langle \mathcal{A}_{(s,\bar{s})}(z)\bar{\mathcal{A}}_{(s,\bar{s})}(z')\right\rangle = f_{\mathcal{A}}(g^2, N, h, \bar{h})\Big\{&\frac{1}{2}D^\alpha\bar{D}^2D_\alpha + \frac{w}{4(\Delta_0+\gamma)}[D^\alpha, \bar{D}^{\dot{\alpha}}]i\partial_{\alpha\dot{\alpha}} + \\ &+ \frac{(\Delta_0+\gamma)^2 + w^2 - 2(\Delta_0+\gamma)}{4(\Delta_0+\gamma)(\Delta_0+\gamma-1)}\Box\Big\}\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(\Delta_0+\gamma)}}\end{aligned} \tag{4.3}$$

where $\Delta_0 = s + \bar{s}$ is the tree-level dimension of the operator, $\omega = s - \bar{s}$ is its R-symmetry charge[2] and γ is the exact anomalous dimension.

The relation (4.3) can be straightforwardly applied to our primary operators $\mathcal{U}_J$. The analysis of the two-point correlator for the $\mathcal{O}_J$'s is somewhat subtler since, as we see from

[2]We assume ω not to renormalize. In fact, once the R-symmetry of the elementary fields is fixed by requiring the exact R-symmetry of the superpotential, any composite operator has a fixed charge given by the sum of the charges of its elementary constituents.

eq. (4.1), these chiral operators are not primaries and in principle the relation (4.3) cannot be applied to their correlators. However, as we are going to show, in the large J limit these operators turn out to behave as CPO's and (4.3) can be safely used.

To this end we remind that in general given a *chiral* operator $\mathcal{A}$, the condition for the operator to be non-protected (anomalous dimension acquired) is equivalent to the condition that its chiral nature is not maintained under superconformal transformations, i.e. $\bar{D}(\delta_{\bar{S}}\mathcal{A}) \sim \{\bar{S}, \bar{D}\}\mathcal{A} \neq 0$ (see for instance [21]). In fact, writing schematically the superconformal algebra relation for a scalar operator as $\{\bar{S}, \bar{D}\} = \Delta - \omega$, we have

$$\{\bar{S}, \bar{D}\}\mathcal{A} = (\Delta - \omega)\mathcal{A} = [(\Delta_0 + \gamma) - \omega]\,\mathcal{A} = \gamma\mathcal{A} \tag{4.4}$$

where we have used $\Delta = \Delta_0 + \gamma$ and for a chiral operator $\omega = \Delta_0$. Therefore if $\gamma \neq 0$, $\bar{S}\mathcal{A}$ is not chiral anymore. Viceversa, if $\{\bar{S}, \bar{D}\}\mathcal{A} = 0$, then $\bar{S}\mathcal{A}$ is still chiral and the dimension is protected by the well-known condition $\Delta = \omega$.

An alternative proof goes through the simple observation that the conditions

$$s + \bar{s} = \Delta_0 + \gamma \qquad s - \bar{s} = \Delta_0 \tag{4.5}$$

imply

$$s = \Delta_0 + \frac{\gamma}{2} \qquad \bar{s} = \frac{\gamma}{2} \tag{4.6}$$

The appearance of $\bar{s} \neq 0$ signals the lack of chirality of the quantum operator.

We now apply the previous argument to our operators O_J to prove that in the large-N, large J limit the violation of chirality is suppressed and they behave as CPO's. In the limit of large R-symmetry $\omega = J$ it is more natural to consider

$$\frac{1}{J}\{\bar{S}, \bar{D}\}\mathcal{O}_J = \left(\frac{\Delta_0 + \gamma}{J} - 1\right)\mathcal{O}_J = \frac{\gamma}{J}\mathcal{O}_J\,. \tag{4.7}$$

As discussed in the previous section, at any fixed order in perturbation theory the anomalous dimension γ does not grow with J. It follows that in the large J limit the r.h.s. of eq. (4.7) is suppressed and the operator behaves as a chiral primary. In particular, in this limit it is consistent to apply eq. (4.3) for the evaluation of its two-point function.

Supported by these considerations we can now proceed exactly as in [13] and find

$$\begin{aligned}
\left\langle \bar{D}^2\mathcal{U}_J(z) D^2\bar{\mathcal{U}}_J(z')\right\rangle &= \frac{N^{J+1}}{(4\pi^2)^{J+1}} f\bar{D}^2\left\{\frac{1}{2}D^\alpha\bar{D}^2D_\alpha + \frac{J-1}{4(J+1+\gamma)}[D^\alpha, \bar{D}^{\dot{\alpha}}]i\partial_{\alpha\dot{\alpha}} + \right.\\
&\qquad \left. + \frac{(J+1+\gamma)^2 + (J-1)^2 - 2(J+1+\gamma)}{4(J+1+\gamma)(J+\gamma)}\Box\right\} \times \\
&\qquad \times D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+1+\gamma)}} \\
&= \frac{N^{J+1}}{(4\pi^2)^{J+1}} f(\gamma^2 + 2\gamma)\bar{D}^2D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\gamma)}}
\end{aligned} \tag{4.8}$$

and

$$\left\langle \mathcal{O}_{J+1}(z)\bar{\mathcal{O}}_{J+1}(z')\right\rangle = \frac{N^{J+2}}{(4\pi^2)^{J+2}} f\bar{D}^2D^2\frac{\delta^4(\theta-\theta')}{|x-x'|^{2(J+2+\gamma)}} \tag{4.9}$$

where f is the common normalization function not fixed by superconformal invariance.

JHEP11(2005)024

From the relation (4.1) the two correlators are related by

$$\langle \bar{D}^2 \mathcal{U}_J(z) D^2 \bar{\mathcal{U}}_J(z') \rangle = |h|^2 \Big| q - \frac{1}{q} \Big|^2 \langle \mathcal{O}_{J+1}(z) \bar{\mathcal{O}}_{J+1}(z') \rangle \tag{4.10}$$

Therefore, inserting in (4.10) the explicit expressions (4.8) and (4.9) we end up with an algebraic equation

$$\gamma^2 + 2\gamma = |h|^2 \Big| q - \frac{1}{q} \Big|^2 \frac{N}{4\pi^2} \tag{4.11}$$

which allows to find the exact expression for the anomalous dimensions

$$\begin{aligned} \gamma &= -1 + \sqrt{1 + |h|^2 \Big| q - \frac{1}{q} \Big|^2 \frac{N}{4\pi^2}} \\ &= \frac{1}{2} |h|^2 \Big| q - \frac{1}{q} \Big|^2 \frac{N}{4\pi^2} - \frac{1}{8} |h|^4 \Big| q - \frac{1}{q} \Big|^4 \frac{N^2}{(4\pi^2)^2} + \cdots . \end{aligned} \tag{4.12}$$

Up to the second order this expression coincides with the perturbative results obtained in the previous section.

We note that our operators $\mathcal{O}_J$ can be thought as dual to the 0-modes of the BMN sector considered in [2, 6, 8, 9]. Formula (4.12) is in agreement with the results presented in those papers for the spectrum of the 0-modes.

5. The superconformal condition at large-N

In the previous section, exploiting the superconformal invariance of the theory and its equations of motion we have shown that the exact anomalous dimension for the $\mathcal{O}_J$ operator can be written as

$$\gamma = -1 + \sqrt{1 + 2\gamma^{(1)}} \tag{5.1}$$

where $\gamma^{(1)}$ is the one-loop anomalous dimension. A direct calculation provides an expression for $\gamma^{(1)}$ proportional to $|h|^2$. As discussed in section 3, at this order we are allowed to use the planar superconformal condition $|h|^2 = g^2$ to re-express $\gamma^{(1)}$ in terms of g^2 only (see eq. (3.7)). Now if in (5.1) we use $\gamma^{(1)}$ given in terms of g^2 and expand the square root, we obtain a perturbative formula for γ which agrees with the actual perturbative calculation only if the condition $|h|^2 = g^2$ is valid at any order.

Motivated by this observation we are led to conjecture that in the large-N limit the condition $|h|^2 = g^2$ is indeed the correct condition for superconformal invariance at any order in perturbation theory. Direct confirmations of this conjecture can be found in the literature up to order g^6 [3–5]. Now we give an argument to prove that this is true to all orders.

We remind that in $\mathcal{N} = 1$ supersymmetric theories the superconformal invariance condition (i.e. vanishing of beta functions) can be expressed as the vanishing of the anomalous dimensions of the elementary superfields [22, 23, 1, 24]. Therefore, in order to study superconformal invariance, it is sufficient to focus on the divergent corrections to the propagators of the elementary fields.

JHEP11(2005)024

In the β-deformed theory we consider a generic L-loop diagram contributing to the propagator of the Φ_i superfield. The crucial observation is the following: If we prove that at the planar level, as long as $q\bar{q} = 1$, this diagram does not depend on q, then we are sure that $|h|^2 = g^2$ is the exact solution of the superconformal invariance equations. In fact, if it is independent of q, the corresponding perturbative contribution is the same for any deformed theory, independently of the choice of the q-deformation. In particular, it is the same for any deformed theory ($q \neq 1$) and for the underformed one ($q = 1$). Focusing on the undeformed case we can conclude that $|h|^2 = g^2$ is the exact condition for the planar superconformal invariance, since $q = 1$ and $|h|^2 = g^2$ bring us back to the $\mathcal{N} = 4$ case which is known to be exactly superconformal. The independence of the perturbative corrections on q allows to extend this statement to any deformed theory.

To conclude the proof we need to show that the contribution from a generic self-energy planar diagram never depends on q. We can focus on diagrams containing only matter vertices because adding vector propagators cannot introduce any q-dependence. We exploit the formal analogy between the deformed theory and noncommutative (nc) field theory. As observed in [2] the deformed potential can be written as

$$ih \int d^6z \, \mathrm{Tr}(\Phi_1 * [\Phi_2, \Phi_3]_*) + \text{h.c.} \tag{5.2}$$

where

$$f * g = e^{i\pi\beta Q_f^{(i)} M_{ij} Q_g^{(j)}} f \cdot g \,, \tag{5.3}$$

$Q^{(i)}$, $i = 1, 2$ being the non-R–symmetry $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ charges and M the antisymmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. When drawing a Feynman diagram we can consider the flow of the charges inside the diagram. Observing that the charges are conserved at any vertex and propagate through the straight lines we can formally identify them with the ordinary momenta in noncommutative diagrams. A known property of planar diagrams in nc field theory is that the star product phase factors dependent on the loop momenta cancel out (for a proof see [25 − 27]) and only an overall phase depending on the external momenta survives. In our case, exploiting the formal identification of charges with momenta, we can use the same arguments to conclude that any planar diagram will have a phase factor from (5.3) depending only on the configuration of the external charges. In the particular case of self-energy diagrams the overall phase is zero since Φ_i and $\bar{\Phi}_i$ have equal (but opposite) charges. In other words, any self-energy planar diagram always contains an equal number of $q = 1/\bar{q}$ and $\bar{q} = 1/q$ vertices. This concludes the proof of the q independence of perturbative self-energy corrections.

We state that in the $N \to \infty$ limit the exact condition for superconformal invariance is simply $|h|^2 = g^2$. Therefore, the theory described by the action

$$\begin{aligned} S = & \int d^8z \, \mathrm{Tr}\left(e^{-gV} \bar{\Phi}_i e^{gV} \Phi^i\right) + \frac{1}{2g^2} \int d^6z \, \mathrm{Tr}\, W^\alpha W_\alpha + \\ & + ig \int d^6z \, \mathrm{Tr}(e^{i\pi\beta} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_1 \Phi_3 \Phi_2) + \\ & + ig \int d^6\bar{z} \, \mathrm{Tr}(e^{i\pi\beta} \bar{\Phi}_1 \bar{\Phi}_2 \bar{\Phi}_3 - e^{-i\pi\beta} \bar{\Phi}_1 \bar{\Phi}_3 \bar{\Phi}_2) \end{aligned} \tag{5.4}$$

represents a $\mathcal{N} = 1$ superconformal invariant theory for any value of β real.

6. Conclusions

For the SU(N), β-deformed $\mathcal{N}=4$ SYM theory we have considered the particular class of operators $\mathcal{O}_J = \mathrm{Tr}(\Phi_1^J \Phi_2)$. We have computed perturbatively their anomalous dimensions up to two loops. The calculation has been performed in the large-N limit in order to avoid mixing with multi-trace operators. Exploiting the techniques introduced in [13] for the BMN operators we have evaluated their *exact* anomalous dimensions in the large-N, large J limit. In this limit the exact expression for the anomalous dimension depends on the deformation parameter q only through the combination $|q-1/q|^2$ and it is then invariant under $q \to -1/q$. Observing that in β-deformed theory exchanging $q \to -1/q$ amounts to exchange $\Phi_i \leftrightarrow \Phi_j$, for any $i = 1,2,3,\ i \neq j$, we may conclude that in the $\mathcal{O}_J$ sector, in the large-N, large J limit there is an enhancement of the SU(3) symmetry and all the operators of the form $\mathrm{Tr}(\Phi_i^J \Phi_k)$, $i \neq k$ renormalize in the same way.

A comparison between the exact result and the perturbative calculation suggests that the condition $|h|^2 = g^2$, which up to three-loops guarantees the superconformal invariance of the theory in the planar limit, is actually sufficient for the exact invariance for $N \to \infty$. Indeed, we have given a direct proof at any order in perturbation theory. The main result of our paper is that the action in (5.4) is superconformal invariant at the quantum level without additional conditions on the couplings. In the context of the AdS/CFT correspondence [28−30] this is the theory whose strong coupling phase is described by the supergravity dual found in [2]. The $\mathcal{O}_J$ sector of this theory for large J shares many similarities with the BMN sector of the $\mathcal{N}=4$ theory in the pp-wave limit. This opens the possibility for these operators to be dual to superstring states in some particular sector of the theory.

It is interesting to consider the extension of our calculations to the case of β complex ($q\bar{q} \neq 1$). In this case the condition for superconformal invariance up to three loops, in the planar limit becomes

$$\frac{1}{2}|h|^2 \left(q\bar{q} + \frac{1}{q\bar{q}} \right) = g^2 \,. \tag{6.1}$$

When (6.1) holds it is easy to see that the perturbative anomalous dimension still coincides with the expansion of the exact result (4.12). As for the case of β real consistency of the perturbative calculation with the exact result would suggest that the condition (6.1) should be valid at any order. However, the proof we have presented in section 5 makes repeated use of the requirement $q\bar{q} = 1$ and cannot be immediately extended to the more general case. A different procedure should be found to prove or disprove that the one-loop condition (6.1) is sufficient to insure the exact conformal invariance even in the case of β complex.

Generalizations of the present results to other deformed theories are presently under investigation and will be reported in [31].

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JHEP11(2005)024

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JHEP11(2005)024